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Flavour violation with a single generation

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Abstract: We calculate probabilities of flavour violating processes mediated by Kalutza-Klein modes of gauge bosons in a model where three generations of the Standard Model fermions arise from a single generation in (5+1) dimensions. We discuss a distinctive feature of the model: while the processes in which the generation number G changes are strongly suppressed, the model is constrained by those with $\Delta G = 0$, for instance $K \to \mu^{\pm} e^{\mp}$. The bound on the size of the extra dimensions is $1/R \gtrsim 64\,\text{TeV}$.

Keywords: Extra Large Dimensions, Quark Masses and SM Parameters, Rare Decays.

Contents

1. Introduction

In models with more than four space-time dimensions, some of long-standing problems of particle physics acquire elegant solutions (see ref. [1] for a review). In particular, in the frameworks of "large extra dimensions" [2], the models have been suggested [3, 4] and studied [5, 6] where three generations of the Standard Model fermions appear as three zero modes localized in the four-dimensional core of a defect with topological number three. When both fermions and Higgs boson are localized on the brane, the overlaps of their wave functions may result in a hierarchical pattern of fermion masses and mixings [7]. This occurs naturally in the models under discussion [3]. To incorporate four-dimensional gauge fields, a compactified version of the model has been developed [8]. There, fermions and scalar fields are localized in the core of a (5+1)-dimensional vortex with winding number three, and two extra dimensions form a sphere accessible for (non-localized) gauge bosons. The zero modes of the gauge bosons are independent from coordinates on the sphere, while higher modes have non-trivial profiles, and hence different overlaps with fermionic zero modes. Since in our model three four-dimensional families appear from a single six-dimensional generation, one can expect flavour violation in the effective four-dimensional theory. Here, we study the specific pattern of these flavour-violating effects, which could distinguish the models of this class from other extra-dimensional models by signatures in rare processes at low energies.

In section 2, we briefly review the model of ref. [8] and discuss the decomposition of gauge fields on a six-dimensional manifold. The couplings of the gauge modes to fermions are calculated in section 3. We study specific flavour-violating processes in section 4 and conclude in section 5 with a description of signatures specific for the given class of models. In appendices, the notations are summarized together with technical details and explicit formulae required for calculations.

2. Gauge bosons on $M^4 \times S^2$

We study the model initially formulated in ref. [3] and developed with a simpler field content in ref. [4]. The model has been compactified on $M^4 \times S^2$, a product of our four-dimensional Minkowski space and a two-dimensional sphere, in ref. [8] (see appendix A for notations). In what follows we will argue that the choice of the manifold is not important for our principal conclusions. The extra dimensions can even be infinitely large, as, for instance, in ref. [9], where well localized gauge-boson zero modes appear. In this case, the role of the radius R of the S^2 sphere is taken by a typical size of the localized gauge zero modes but not by a size of extra dimensions.

The interaction of vector-like counterparts of the fermions of one Standard Model generation with the vortex field of the abelian Higgs model results in k chiral zero modes of each fermion localized in the four-dimensional core of the vortex; k is the winding number of the vortex and is equal to three in our case. These three zero modes of a single six-dimensional fermion represent the corresponding four-dimensional fermions of three generations; the zero modes are linearly independent and hence have different windings in φ. Detailed descriptions of the model can be found in refs. [3, 4, 8]; here, we only outline the setup and introduce some notations.

The vortex is formed by a scalar field, which extends to a typical size $R\theta_\Phi$ from the origin, and a gauge field of size $R\theta_A$. Apart from these two fields whose non-trivial profiles have a topological origin, there is also the Standard Model Higgs doublet H which, due to the interaction with the vortex scalar field, also develops a non-trivial profile (see ref. [10]): it is non-zero inside the core of the vortex, and its vacuum expectation value vanishes in the bulk. The typical size of H is also $R\theta_\Phi$. Due to the interaction with H, the fermionic zero modes, whose size is $R\theta_A$, aquire small (as compared to the energy scale of the vortex) masses. The hierarchical fermionic mass pattern is governed by a small parameter [8]

$$\delta = \frac{\theta_\Phi}{\theta_A} \sim 0.1\,.$$

In what follows we will also assume that $\theta_\Phi < \theta_A \ll 1$.

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The gauge bosons of the Standard Model SU(3) $\times$ SU(2) $\times$ U(1) group do not interact directly with the vortex field.[1] The interaction of the SU(2) $\times$ U(1) bosons with the Higgs doublet H ensures the proper pattern of the electroweak symmetry breaking inside the core of the vortex, that is in our usual four-dimensional space.

Let us perform a Kaluza-Klein decomposition of a gauge field. We start with the U(1) gauge field $\mathcal{A}_A$ whose action on $M^4 \times S^2$ is

$$S = -\frac{1}{4}\int d^6X\,\sqrt{-G}\,G^{AC}G^{BD}\mathcal{F}_{AB}\mathcal{F}_{CD}\,,$$

where $\mathcal{F}_{AB} = \partial_A\mathcal{A}_B - \partial_B\mathcal{A}_A$. The separation of variables in the equations of motion is straightforward in the gauge

$$\begin{aligned}\partial_\mu\mathcal{A}^\mu &= 0\,,\\ \frac{1}{\sin\theta}\partial_\theta(\sin\theta\mathcal{A}_\theta) + \frac{1}{\sin^2\theta}\partial_\varphi\mathcal{A}_\varphi &= 0\end{aligned}$$

and results in the four-dimensional effective lagrangian for the Kaluza-Klein modes,

$$\mathcal{L} = \mathcal{L}_V + \mathcal{L}_S\,,$$

where

$$\begin{aligned}\mathcal{L}_V &= \frac{1}{2}A_\nu\partial_\mu^2A^\nu + \frac{1}{2}\sum_{l=1}^{\infty}A_{l,\nu}\left(\partial_\mu^2 + \frac{l(l+1)}{R^2}\right)A_l^\nu + \\ &\quad+\sum_{l=1}^{\infty}\sum_{m=1}^{l}A^*_{(l,m)\nu}\left(\partial_\mu^2 + \frac{l(l+1)}{R^2}\right)A^\nu_{(l,m)}\,, \qquad (2.1)\\ \mathcal{L}_S &= -\frac{1}{2}\sum_{l=1}^{\infty}B_l\left(\partial_\mu^2 + \frac{l(l+1)}{R^2}\right)B_l - \sum_{l=1}^{\infty}\sum_{m=1}^{l}B^*_{l,m}\left(\partial_\mu^2 + \frac{l(l+1)}{R^2}\right)B_{l,m}\,. \qquad (2.2)\end{aligned}$$

The only massless gauge field A_μ, Kaluza-Klein vector fields $A_{l,\nu}$, $A_{(l,m),\nu}$ and massive scalar fields B_l, $B_{l,m}$ are defined in appendix B. The massless mode A_μ represents the four-dimensional gauge field which depends neither on θ nor on φ in our case.

We turn now to the non-abelian gauge bosons. For the unbroken gauge symmetry case, the quadratic lagrangian of a non-abelian vector field reproduces eqs. (2.1), (2.2). Non-observation of Kalutza-Klein modes at colliders implies that the size R of extra dimensions should be significantly smaller than inverse Z-boson mass. This fact allows one to treat the electroweak symmetry breaking perturbatively. In this approach, the impact of the background Higgs field $H(\theta)$, which is localized in the core of the vortex (see ref. [4]) for details), on the eigensystem of gauge modes, is considered as a small perturbation compared to curvature of the sphere. We demonstrate in appendix C that the lowest modes of $W^\pm$ and Z bosons aquire proper masses in this way. The corresponding eigenfunction is no longer constant. However, since the Higgs background, $H(\theta)$, is independent of φ, the

[1] We neglect here possible kinetic mixing between the hypercharge and vortex gauge bosons which is irrelevant for flavour violation.

perturbed mode does not depend on φ as well. As we will see in appendix D, it means that this mode does not mediate flavour-changing processes. Contrary to the unbroken case, the θ dependence of the lowest mode results in different couplings of the fermions of three generations to $W^{\pm}$ and Z bosons, given different localizations of the fermions in extra dimensions. This effect, however, is suppresed as compared to the family non-universal interactions of fermions with non-zero modes of gauge bosons.

3. Coupling of gauge modes to fermions

The interaction of a six-dimensional fermion Ψ with the six-dimensional photon $\mathcal{A}_A$ is determined by the following term in the lagrangian,

$$\mathcal{L}_6 = \sqrt{-G}\, e_6 \mathcal{A}_A \bar{\Psi} \Gamma^A(X) \Psi \equiv \sqrt{-G}\, e_6 \mathcal{A}_A J^A \,, \tag{3.1}$$

where coordinate-dependent Dirac matrices $\Gamma^A(X)$ are defined in appendix A. In the zero-mode approximation,

$$\Psi(X) = \sum_{n=1}^{3} a_n(x) \otimes \begin{pmatrix} 0 \\ f_2(\theta, n) \cdot \mathrm{e}^{i\varphi \frac{7-2n}{2}} \\ f_3(\theta, n) \cdot \mathrm{e}^{i\varphi \frac{1-2n}{2}} \\ 0 \end{pmatrix} , \tag{3.2}$$

where $a_n(x)$, $n = 1, 2, 3$, are three four-dimensional two-component spinors which represent three generations of fermions with quantum numbers of Ψ (see refs. [3, 8] for details). The functions f_i are normalized as

$$2\pi \int_0^{\pi} d\theta \sqrt{-G}\, [f_2(n)^2 + f_3(n)^2] = 1 \,. \tag{3.3}$$

It is important to note that the angular momentum in transverse dimensions (that is, the winding number of a wave function) corresponds to the number of generation: the index n in (3.2) enumerates the families. As a result, this number is conserved in the first approximation (this symmetry is broken by the terms responsible for inter-generation mixing, see ref. [4]). As we will see below, this feature results in unusually strong suppression of many flavour-violating processes.

The effective four-dimensional fermion-gauge lagrangian is calculated in appendix D. It can be conveniently rewritten as

$$\mathcal{L}_4 = e \cdot \mathrm{Tr}(\mathbf{A}^{\mu} \mathbf{j}^{*}{}_{\mu}) \,, \tag{3.4}$$

where

$$\mathbf{A}^{\mu} = (\mathbf{A}^{\mu})^{\dagger} = \sum_{l=0}^{\infty} \begin{pmatrix} E_{11}^{l,0} A_{l,0}^{\mu} & E_{12}^{l,1} A_{l,1}^{\mu} & E_{13}^{l,2} A_{l,2}^{\mu} \\ E_{21}^{l,1} A_{l,1}^{\mu *} & E_{22}^{l,0} A_{l,0}^{\mu} & E_{23}^{l,1} A_{l,1}^{\mu} \\ E_{31}^{l,2} A_{l,2}^{\mu *} & E_{32}^{l,1} A_{l,1}^{\mu *} & E_{33}^{l,0} A_{l,0}^{\mu} \end{pmatrix} , \tag{3.5}$$

$$j_{mn}^{\mu}(x) = a_m^{\dagger} \bar{\sigma}^{\mu} a_n \,,$$

and $e \equiv e_6/\sqrt{4\pi} R$ is the usual four-dimensional coupling.

The coupling constants $E_{mn}^{l,n-m}$ are defined and estimated in appendix D,

$$E_{mn}^{l,m-n} \sim \begin{cases} l^{|m-n|+1/2}\theta_A^{|m-n|} & \text{at } \; l\theta_A \ll 1\,, \\ \dfrac{1}{\sqrt{\theta_A}} & \text{at } \; l \simeq \dfrac{1}{\theta_A}\,, \\ \mathrm{e}^{-lF(\theta_A)} & \text{at } \; l\theta_A \gg 1\,. \end{cases} \tag{3.6}$$

We see that the fermions have strongest couplings to the heavy modes with masses

$$m_l = \frac{\sqrt{l(l+1)}}{R} \sim \frac{1}{\theta_A R}\,.$$

The reason for this is obvious: modes with $l \sim 1/\theta$ have largest overlaps with fermionic wavefunctions of the size θ ($\theta \approx \theta_A$ in our case); (lower modes have larger width in θ while higher modes oscillate several times at the width of the fermions). We stress that this feature depends neither on details of localization of fermions and gauge bosons nor on the shape and size of extra dimensions.

The fermions a_n, which enter the current $\mathbf{j}_\mu$ and consequently appear in the lagrangian (3.4), are the states in the gauge basis, while physically observed mass eigenstates are their linear combinations. In particular, the mass matrix of the fermions with quantum numbers of the down-type quarks is given [4, 8] by

$$M_D = \begin{pmatrix} m_{11} & m_{12} & 0 \\ 0 & m_{22} & m_{23} \\ 0 & 0 & m_{33} \end{pmatrix} \propto \begin{pmatrix} \delta^4 & \epsilon\delta^3 & 0 \\ 0 & \delta^2 & \epsilon\delta \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.7}$$

where

$$\delta = \frac{\theta_\Phi}{\theta_A} \sim \sqrt[4]{\frac{m_{11}}{m_{33}}} \sim \sqrt[4]{\frac{m_b}{m_d}} \sim 0.1\,,$$
$$\epsilon \sim 0.1\,.$$

To diagonalize the mass matrix one should use biunitary transformations,

$$S_d^\dagger M_D T_d = M_D^{\mathrm{diag}}\,.$$

The fermions in the mass basis are

$$Q_n = (S_d^\dagger)_{nm} q_m\,, \qquad D_n = (T_d^\dagger)_{nm} d_m\,,$$

where we denoted a_n as q_n for the left-handed and as d_n for the right-handed down-type quarks. If one rewrites the current $\mathbf{j}_\mu$ in terms of the mass eigenstates, then the matrix $\mathbf{A}^\mu$, eq. (3.5), should be replaced by

$$\tilde{\mathbf{A}}^\mu = S_d^\dagger \mathbf{A}^\mu S_d\,.$$

Explicit expressions for S, T and $\tilde{\mathbf{A}}^\mu$ are given in appendix E.

The interaction of fermions with $W^\pm$ and Z bosons is very similar to the electromagnetic couplings discussed above. There are two differences: firstly, the current $\mathbf{j}_\mu$ in eq. (3.4) is replaced by the Standard-Model charged and neutral weak currents; secondly, the gauge eigensystem is modified as discussed in appendix C. The latter modification does not change the results significantly: it is negligible for Kalutza-Klein modes and it does not result in flavour violation for the lowest mode.

To confront the model with the experimental results, one needs to calculate the effective four-fermion coupling g_{mn}, that is, in each particular case, to sum up the contributions

$$g_{mn}^l = e^2 \frac{(E_{mn}^{l,m-n})^2}{m_l^2}$$

for all l. A very naive estimate gives, using eq. (3.6),

$$g_{mn} \sim e^2 l_{\max} \cdot R^2 \theta_A = \frac{e^2}{\theta_A} \cdot R^2 \theta_A = e^2 R^2 \tag{3.8}$$

so that

$$\frac{g_{mn}}{G_F} \sim (M_W R)^2 \,.$$

This result is supported by more explicit calculations given in appendix F.

4. Flavour violating processes

We turn now to the study of specific flavour violating processes which are known to give the strongest constraints on masses and couplings of new vector bosons. The most stringent bounds arise [11] from K_L^0 — K_S^0 mass difference, forbidden K decays $K_L^0 \to \mu e$, $K^+ \to \pi^+ e^- \mu^+$ (see also ref. [12]) and from lepton flavour violating processes $\mu \to e\gamma$, $\mu \to 3e$ and $\mu \to e$ conversion on a nuclei. We discuss all these constraints below.

First of all, one should note that without account of inter-generation mixings, the generation number G is exactly conserved. Indeed, the integration over ϕ in the effective lagrangian results in the corresponding selection rules: in eq. (3.5), no vector boson has both diagonal and off-diagonal couplings simultaneously. This forbids all processes with nonzero change of G; the probabilities of the latters in the full theory are thus suppressed by powers of the mass-matrix mixing parameter, $(\epsilon\alpha)^{\Delta G}$ (α is determined in appendix E). However, the amplitudes of processes with $\Delta G = 0$ but lepton and quark flavours violated separately are suppressed only by the mass squared of the Kalutza-Klein modes. The best studied among these processes are kaon decays $K_L^0 \to \mu e$ and $K^+ \to \pi^+ e^- \mu^+$, forbidden in the Standard Model with massless neutrinos because of separate conservation of e and μ lepton numbers.[2] In the rest of this section, we estimate, in the frameworks of the full theory with mixing, the size of flavour-violating effects for different values of ΔG.

[2] Amplitude of the $K_L^0 \to \mu e$ process due to non-zero neutrino masses is thirty orders of magnitude smaller than the best experimental limit [13].

4.1 $\Delta G = 0$: forbidden kaon decays

The best experimental restriction on flavour-violating processes with $\Delta G = 0$ is the branching ratio of $K_L^0 \to \mu e$ decay [14],

$$\mathrm{Br}(K_L^0 \to \bar{\mu}^+ e^-) < B = 2.4 \cdot 10^{-12} \,. \tag{4.1}$$

The K^0 meson is a pseudoscalar, and the decay cannot be mediated by purely vector interaction of the Kalutza-Klein modes of the photon. However, the higher modes of the Z boson interact with a $V - A$ current and contribute to the decay width. From eq. (E.2) one obtains, in particular, the dominant, unsuppressed by $(\epsilon\delta)$, axial coupling in the four-dimensional lagrangian,

$$\frac{g}{2\cos\theta_W} \sum_{l=1}^{\infty} E_{1\,2}^{l,1} Z_{l,1}^{\mu} \left(-\frac{1}{2}\bar{s}\gamma_\mu\gamma_5 d - \frac{1}{2}\bar{e}\gamma_\mu\gamma_5\mu - (\frac{1}{2} - 2\sin^2\theta_W)\bar{e}\gamma_\mu\gamma_5\mu \right) .$$

The diagrams for this and other processes are similar to those given in [11, 12]; one has to sum over all intermediate Kalutza-Klein modes to obtain the effective four-fermion coupling in a way similar to section 3 or appendix F,

$$\sum_{l=1}^{\infty} \frac{(E_{1\,2}^{l,1} R)^2}{l(l+1)} = \zeta R^2 \,,$$

where $\zeta \approx 0.4$ is a coefficient which results from numerical evaluation of the sum.

The partial width of the $K_L^0 \to \mu e$ decay is easy to estimate by comparison to the branching ratio of $K^+ \to \mu^+\nu$: in the $m_e \ll m_\mu$ approximation, the phase volume and f_K factors cancel in the width ratio,

$$\mathrm{Br}(K_L^0 \to \mu e) = \frac{\Gamma(K_L^0 \to \mu e)}{\Gamma(K_L^0 \to \mathrm{all})} = \mathrm{Br}(K^+ \to \mu^+\nu) \frac{\tau(K_L^0)}{\tau(K^+)} \left| \frac{\langle \bar{\mu} e | \bar{s} d \rangle}{\langle \bar{\mu}\nu | \bar{s} u \rangle} \right|^2 ,$$

where $\tau(K_L^0)$ and $\tau(K^+)$ are the lifetimes of the corresponding particles. The interaction responsible for the $K^+ \to \mu^+\nu$ decay in the Standard Model is

$$\frac{g}{2\sqrt{2}} W^\mu \left(\bar{s}\gamma_\mu(1+\gamma_5) u \sin\theta_c + \bar{\mu}\gamma_\mu(1+\gamma_5)\nu_\mu \right) , \tag{4.2}$$

where θ_c is the Cabibbo angle, and results in the four-fermionic matrix element,

$$\langle u\bar{s} | \bar{\mu}\nu \rangle_t = \frac{2g^2 \sin\theta_c}{8M_W^2} \,. \tag{4.3}$$

Hereafter, we denote as $\langle \cdots \rangle_t$ a matrix element with truncated wave functions and spinorial structure. Using ref. [19] we obtain theoretical prediction on the branching ratio of the process $K_L^0 \to \mu^+ e^-$, from which one obtains the following bound on the size of the sphere,

$$\frac{1}{R} > \frac{M_W}{\cos\theta_W} \left(\frac{\zeta}{\sin\theta_C} \right)^{1/2} \left(\frac{\mathrm{Br}(K^+ \to \mu^+\nu)}{B} \, \frac{\tau(K_L^0)}{\tau(K^+)} \, \frac{2\sin^4\theta_W - \sin^2\theta_W + 1/4}{2} \right)^{1/4} . \tag{4.4}$$

We use all necessary numerical values from ref. [14] and obtain the restriction,

$$\frac{1}{R} > 101\sqrt{\zeta}\,\mathrm{TeV} \approx 64\,\mathrm{TeV}\,.$$

In a similar way, the width of the decay $K^+ \to \pi^+\mu^+e^-$ can be compared to one of $K^+ \to \pi^0\mu^+\nu$. The relevant interactions are

$$e\sum_{l=1}^{\infty} E_{1\,2}^{l,1} A_{l,1}^{\mu}\left\{-\frac{1}{3}\bar{s}\gamma_\mu d + \bar{e}\gamma_\mu\mu\right\} +$$
$$+\frac{g}{2\cos\theta_W}\sum_{l=1}^{\infty} E_{1\,2}^{l,1} Z_{l,1}^{\mu}\left\{\left(\frac{2}{3}\sin^2\theta_W - \frac{1}{2}\right)\bar{s}\gamma_\mu d - \bar{e}\left(2\sin^2\theta_W - \frac{1}{2} + \frac{1}{2}\gamma_5\right)\gamma_\mu\mu\right\}.$$

For the decay $K^+ \to \pi^0\mu^+\nu$, the relevant interaction is given by eq. (4.2), and the matrix element $\langle\bar{u}\bar{\mu}\nu|\bar{s}\rangle_t$ coincides with eq. (4.3). Together with the limit [14]

$$\mathrm{Br}(K^+ \to \pi^+\mu^+e^-) < B_1 = 2.8\cdot 10^{-11}\,,$$

this determines that

$$\frac{1}{R} > \frac{M_W}{\cos\theta_W}\left(\frac{\zeta}{2\sin\theta_C}\right)^{1/2}\left(\frac{\xi\,\mathrm{Br}(K^+ \to \pi^0\mu^+\nu)}{B_1}\right)^{1/4},$$

where

$$\xi = \left(\frac{4\sin^2\theta_W}{3} - 1\right)^2 (1 + (4\sin^2\theta_W - 1)^2) + \frac{\sin^4\theta_W}{9}(16\cos^2\theta_W)^2\,.$$

So, constraint from this decay is

$$\frac{1}{R} > 25\,\mathrm{TeV}\,,$$

which is less restrictive than eq. (4.4).

4.2 $\Delta G = 1$: lepton flavour violation

As we have already noted, the processes with $\Delta G \neq 0$ are suppressed by powers of the mixing parameter $\epsilon\alpha$. Indeed, it follows from eq. (E.2) that these processes could be mediated by "diagonal" vector bosons $A_{l,0}^{\mu}$, and each corresponding diagram contains one vertex suppressed by $\epsilon\alpha$.

The analysis of the $\mu \to ee\bar{e}$ process is very similar to one of kaon decays. The interaction terms are

$$e\sum_{l=1}^{\infty} E_{1\,1}^{l,0} A_{l,0}^{\mu}\bar{e}\gamma_\mu\left[(\epsilon_L\alpha_L)\mu + e\right] +$$
$$+\frac{g}{2\cos\theta_W}\sum_{l=1}^{\infty} E_{1\,1}^{l,0} Z_{l,0}^{\mu}\left[\bar{e}\gamma_\mu\left(\left[2\sin^2\theta_W - \frac{1}{2}\right] - \frac{1}{2}\gamma_5\right)((\epsilon_L\alpha_L)\mu + e)\right],$$

we denoted the parameters ϵ and α of the leptonic mixing matrix as ϵ_L and α_L. There is also a contribution to this process (as well as to the μe-conversion discussed below) mediated by "off-diagonal" bosons $A^\mu_{l,1}$. The contribution has the same order (suppresed by $\epsilon_L\alpha_L$) and the opposite sign. We do not suspect, however, that some cancellation of the diagrams is present.

Following ref. [19] again, we obtained width of decay

$$\Gamma(\mu\to ee\bar{e}) = \frac{G_F^2 m_\mu^5}{192\pi^3}(m_W R)^4(\epsilon_l\alpha_S)^2(\zeta)^2\frac{1+20\sin^4\theta_W}{2\cos^4\theta_W}\,,$$

so the limit is

$$\frac{1}{R} > 96\sqrt{\epsilon_l\alpha_L\zeta}\,\mathrm{TeV}\,.$$

In the leptonic sector, the mixing parameter ϵ_L is unknown; however even $\alpha_L\sim\delta_L\sim(m_e/m_\tau)^{1/4}$ results in additional suppression.

Traditionally, one of the strongest constraints on the masses and couplings of new vector bosons arises from μe-conversion on nuclei. Let us show that, in our model, this bound is not so restrictive. Adopting the calculation of ref. [15, 16] to our case, we estimate the relative muon conversion rate on a nucleus with charge Z and neutron number N as

$$\mathcal{R}\equiv\frac{\Gamma_{\mathrm{conv}}}{\Gamma_{\mathrm{capt}}} = 2\frac{\alpha^3_{\mathrm{QED}}m_\mu^5}{\pi^2\Gamma_{\mathrm{capt}}}\frac{Z^4_{\mathrm{eff}}}{Z}|F(q)|^2(\epsilon_L\alpha_L)^2\zeta^2R^4\kappa\,,$$

where

$$\kappa = \left(|(2Z+N)\xi_{Lu}+(Z+2N)\xi_{Ld}|^2+|(2Z+N)\xi_{Ru}+(Z+2N)\xi_{Rd}|^2\right)\,,$$

and

$$\begin{aligned}
\xi_{Ld} &= \frac{2\sin^2\theta_W}{3}+\frac{(-1/2+\sin^2\theta_W)(-1/2+2\sin^2\theta_W/3)}{\cos^2\theta_W}\approx 0.275\,;\\
\xi_{Rd} &= \frac{2\sin^2\theta_W}{3}+\frac{\sin^2\theta_W(-1/2+2\sin^2\theta_W/3)}{\cos^2\theta_W}\approx 0.050\,;\\
\xi_{Lu} &= -\frac{4\sin^2\theta_W}{3}+\frac{(-1/2+\sin^2\theta_W)(1/2-4\sin^2\theta_W/3)}{\cos^2\theta_W}\approx -0.374\,;\\
\xi_{Ru} &= -\frac{4\sin^2\theta_W}{3}+\frac{\sin^2\theta_W(1/2-4\sin^2\theta_W/3)}{\cos^2\theta_W}\approx -0.249\,.
\end{aligned}$$

For the titanium nuclei [15], the muon capture rate $\Gamma_{\mathrm{capt}}\approx 2.6\cdot 10^6\ \mathrm{s}^{-1}$, the effective charge $Z_{\mathrm{eff}}\approx 17.6$, the nuclear form factor $|F(q)|\approx 0.54$, $Z=22$, $N=26$ and the strongest limit [14] is

$$\mathcal{R}<\mathcal{R}_1\approx 4.3\cdot 10^{-12}\,.$$

We obtain the constraint on R and $(\epsilon_L A_L)$ from the non-observation of μe-conversion on nuclei,

$$\frac{1}{R} > 124\sqrt{\epsilon_L\alpha_L}\,\mathrm{TeV}\approx 12\,\mathrm{TeV}\,.$$

The bound from $\mu\to e\gamma$ decay is further suppressed by a loop factor.

4.3 $\Delta G = 2$: $K_L - K_S$ mass difference and CP violation in kaons

Non-universal couplings of the gauge bosons would contribute also to the kaon mass difference ($m_{K_L} - m_{K_S}$) which was measured with a good accuracy. They also would induce additional CP-violation effects. In our case, however, these contributions are suppressed.

Indeed, the relevant interaction reads[3]

$$
\begin{aligned}
&-(\epsilon_d\alpha_d)\frac{e}{3}\sum_{l=1}^{\infty}\left(E_{11}^{l,0}-E_{22}^{l,0}\right)A^{\mu}_{l,0}\bar{s}\gamma_\mu d+\\
&+(\epsilon_d\alpha_d)\frac{g}{2\cos\theta_w}\sum_{l=1}^{\infty}\left(E_{11}^{l,0}-E_{22}^{l,0}\right)Z^{\mu}_{l,0}\bar{s}\gamma_\mu\left[\left(\frac{2}{3}\sin^2\theta_w-\frac{1}{2}\right)-\frac{1}{2}\gamma_5\right]d+\\
&+(\epsilon_d\alpha_d)g_s\sum_{l=1}^{\infty}\left(E_{11}^{l,0}-E_{22}^{l,0}\right)G^{\mu i}_{l,0}\bar{s}\gamma_\mu\frac{\lambda_i}{2}d+\text{h.c.}\,,
\end{aligned}
$$

where the last term represents the interaction with the Kalutza-Klein modes of the gluon. The latter contribution dominates over the former two because of larger coupling g_s.

One estimates the contribution to Δm_K from the exchange of the higher modes of the gluon field as

$$
\begin{aligned}
\Delta' m_K &\approx 2\mathrm{Re}\langle K_0|H_{\Delta G=2}|\bar{K}_0\rangle\\
&= m_K f_K^2\frac{8g_S^2}{9}\left\{(\epsilon_d\alpha_d)^2+(\epsilon_u\alpha_u)^2+\left(\frac{m_K}{m_s+m_d}\right)^2\epsilon_d\alpha_d\epsilon_u\alpha_u\right\}\times\\
&\quad\times\sum_{l=1}^{\infty}\left(E_{11}^{l,0}-E_{22}^{l,0}\right)^2\frac{R^2}{l(l+1)}\,,
\end{aligned}
\tag{4.5}
$$

where the matrix element was estimated in the vacuum insertion approximation (see ref. [17] and references therein). We note that, besides the expected $(\epsilon_d\alpha_d)^{\Delta G}$, additional suppression factors arise in the four-fermionic interaction due to the following two reasons. Firstly, $E_{11}^{l,0}\approx E_{22}^{l,0}$: as it is shown in appendix D, $E_{mn}^{l,m-n}$ depends, in the first approximation, on $|m-n|$ and not on m and n separately. In the second approximation, in a way similar to section 3 or appendix F, one obtains

$$
\sum_{l=1}^{\infty}\left(E_{11}^{l,0}-E_{22}^{l,0}\right)^2\frac{1}{l(l+1)}\sim\theta_A^2\sim 0.01\,.
$$

The second reason is that the matrices T_d and S_u are almost diagonal: $\alpha_u\sim\delta^3$ whereas $\alpha_d\sim\delta$ (see appendix E). This suppresses the third term in the curled brackets in eq. (4.5) down to the order of the first one. Therefore, for $g_s\approx 1.1$ and all other parameters from ref. [14], we obtain the following limit on R,

$$
\frac{1}{R}>(\epsilon_d\alpha_d)g_s f_K\theta_A\sqrt{\zeta\frac{8}{9}\left(1+\left(\frac{m_K}{m_d+m_s}\right)^2\frac{\epsilon_u\alpha_u}{\epsilon_d\alpha_d}\right)\frac{m_K}{\Delta m_K}}
$$

[3]The fermionic wave functions in this equation do not include the complex phase factors F_d, $F_d^\dagger$. These factors redefine (reduce) the phase of ϵ_d (see appendix E for details).

$$= (\epsilon_d \alpha_d \theta_A) \sqrt{1 + 30 \frac{\epsilon_u \alpha_u}{\epsilon_d \alpha_d}} \cdot 1300 \,\text{Tev} \approx 1.5 \,\text{Tev} \,.$$

This value is small compared to the restriction from the limit on the branching ratio of $K_L \to \mu e$ decay. However, it has been pointed out in ref. [18] that a stronger bound on the size of extra dimensions may arise due to additional contributions to the CP-violating parameter ε_K. In the five-dimensional model of ref. [18], this restriction is about an order of magnitude stronger than one from Δm_K. This is not the case in a class of models considered here. Indeed,

$$\varepsilon_K = \frac{\mathrm{Im} \langle K_0 | H_{\Delta G=2} | \bar{K}_0 \rangle}{2\sqrt{2} \Delta m_K} \,.$$

The imaginary part arises from the phase of coupling constant of fermions with higher gauge modes (which does not necessarily coincide with the phase of CKM matrix). This phase is also suppressed (see appendix E) due to the same "almost" diagonal structure of the matrix S_u. Taking into account this notion, we obtain the limit

$$\frac{1}{R} > (\epsilon_d \alpha_d) g_s f_K \theta_A \sqrt{\zeta \frac{8}{9} \left(1 + \left(\frac{m_K}{m_d + m_s}\right)^2 \frac{\epsilon_u \alpha_u}{\epsilon_d \alpha_d}\right) \frac{m_K}{\Delta m_K}} \sqrt{\frac{\epsilon_u \alpha_u}{\epsilon_d \alpha_d \sqrt{2} \varepsilon_K}} \approx 2.6 \,\text{Tev} \,.$$

Here, the amplification by $1/\sqrt{\varepsilon_K}$ is present, similarly to ref. [18], but an additional suppression by a factor of $\sqrt{\epsilon_u \alpha_u / \epsilon_d \alpha_d}$ diminishes this effect.

5. Conclusions

The model with a single generation of vector-like fermions in six dimensions allows one to explain fermionic mass hierarchy without introducing a flavour quantum number: three families of four-dimensional fermions appear as three sets of zero modes developed on a brane by a single multi-dimensional family while the fermionic wave functions inevitably produce a hierarchical mass matrix due to different overlaps with the Higgs field profile. The six-dimensional lagrangian with one generation contains much less parameters than the effective one. All masses and mixings of the Standard-Model fermions are governed by a few parameters of order one. This fact allows for specific phenomenological predictions. In particular, in a compactified version of the model with non-localized gauge fields, the Kalutza-Klein modes of the vector bosons mediate flavour-violating processes studied in this paper. The pattern of flavour violation is distinctive: contrary to other extra-dimensional models, processes with change of the generation number G by one or two units are strongly suppressed compared to other rare processes. For example, $\mathrm{Br}(\mu \to \bar{e}ee)/\mathrm{Br}(K \to \mu^\pm e^\mp) \sim 1/10$. The strongest constraint on the model arises from non-observation of the decay $K \to \mu^\pm e^\mp$; it requires that the size of the extra-dimensional sphere (size of the gauge-boson localization) R satisfies $1/R \gtrsim 64\,\text{TeV}$. The Kalutza-Klein modes of vector bosons have larger masses, but for large enough R, could be detected indirectly by precision measurements at future linear colliders. A clear signature of the model would be an observation of $K \to \mu^\pm e^\mp$ decay without observation of $\mu \to \bar{e}ee$, $\mu \to e\gamma$ and μe-conversion at the same precision level.

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A. Notations

The metric G_{AB} of $M^4 \times S^2$ is determined by

$$ds^2 = G_{AB} dX^A dX^B = \eta_{\mu\nu} dx^\mu dx^\nu - R^2(d\theta^2 + \sin^2\theta d\varphi^2)\,,$$

where $\eta_{\mu\nu} = \mathrm{diag}(+,-,\ldots,-)$ is the four dimensional Minkowski metric, capital Latin indices enumerate the coordinates X^A on $M^4 \times S^2$, $A, B = 0, \ldots, 5$; Greek indices refer to the coordinates x_μ on M^4, $\mu, \nu = 0, \ldots, 3$. We reserve lower case Latin indices $a, b = 0, \ldots, 5$ for a flat six-dimensional tangent space. The minimal representation for six-dimensional spinors is eight-component. In the curved space, the Dirac matrices depend on the coordinates,

$$\Gamma^A(X) = h^A_a \Gamma^a\,, \tag{A.1}$$

where the sechsbein in our case is given by

$$h^a_A = (\delta^a_\mu, \delta^a_4 R, \delta^a_5 R\sin\theta) \tag{A.2}$$

and flat space 8×8 Dirac matrices are

$$\Gamma^\mu = \begin{pmatrix} 0 & \begin{matrix} \bar\sigma^\mu & 0 \\ 0 & \sigma^\mu \end{matrix} \\ \begin{matrix} \sigma^\mu & 0 \\ 0 & \bar\sigma^\mu \end{matrix} & 0 \end{pmatrix}, \qquad \Gamma^{4,5} = \begin{pmatrix} 0 & \begin{matrix} 0 & \pm 1 \\ 1 & 0 \end{matrix} \\ \begin{matrix} 0 & \mp 1 \\ -1 & 0 \end{matrix} & 0 \end{pmatrix}, \tag{A.3}$$

where $\sigma_\mu = (1, \sigma_i)$, $\bar\sigma_\mu = (1, -\sigma_i)$, and σ_i $(i = 1, 2, 3)$ are the Pauli matrices.

B. Decomposition of the gauge field

After separation of variables, the six-dimensional U(1) gauge field $\mathcal{A}_A$ is decomposed as

$$\begin{aligned}\mathcal{A}_\mu(X) &= \frac{1}{R}\left(\sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{(l,m)\mu}(x) Y_{lm}(\theta,\phi)\right) \\ &\equiv \frac{1}{R}\left(\frac{A_\mu(x)}{\sqrt{4\pi}} + \sum_{l=1}^{\infty} A_{l,\mu}(x) Y_{l0}(\theta,\phi) + \sum_{l=1}^{\infty}\sum_{m=-l,\,m\neq 0}^{l} A_{(l,m)\mu}(x) Y_{lm}(\theta,\phi)\right),\end{aligned}$$

$$\mathcal{A}_\theta(X) = \frac{1}{\sin\theta}\sum_{l=1}^{\infty}\sum_{m=-l,\, m\neq 0}^{l} B_{l,m}(x)\frac{|m|}{\sqrt{l(l+1)}}Y_{lm}(\theta,\phi)\,,$$

$$\begin{aligned}\mathcal{A}_\varphi(X) &= i\sin\theta\sum_{l=1}^{\infty}\sum_{m\neq 0}\frac{|m|}{m}\frac{1}{\sqrt{l(l+1)}}B_{l,m}(x)\partial_\theta Y_{lm}(\theta,\phi) + \\ &\quad + \sin\theta\sum_{l=1}^{\infty}B_l(x)\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+1)!}{(l-1)!}}P_l^1(\cos\theta)\,,\end{aligned}$$

where $Y_{lm}(\theta,\phi)$ are properly normalized,

$$\int_0^{2\pi}\int_0^{\pi} Y^*_{l'm'}Y_{lm}\sin\theta\, d\varphi\, d\theta = \delta_{l,l'}\,\delta_{m,m'}\,,$$

spherical harmonics,

$$Y_{lm} = (-1)^{\frac{m+|m|}{2}}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-|m|)!}{(l+|m|)!}}P_l^{|m|}(\cos\theta)\mathrm{e}^{im\varphi}\,,$$

and $P_l^m(x)$ are adjoint Legendre functions.

C. Electroweak symmetry breaking

In the six-dimensional theory, only one scalar field, H, is charged under the electroweak group [4]. The soliton-like solution for the Higgs-vortex system breaks the electroweak symmetry, and non-zero values of four-dimensional masses of W and Z bosons arise. The classical Higgs profile is independent of φ, so the lowest modes of massive gauge bosons in the background of the soliton do not depend on φ as well. As a result, the Z boson itself does not mediate flavour changing processes. The masses of the lowest modes can be calculated by means of the perturbation theory in a small parameter g^2. In the zeroth approximation, the eigenfunctions are constant zero modes of the Laplace operator, equal to $1/\sqrt{4\pi}R$. The first-order correction to the lagrangian is

$$\Delta\mathcal{L}_6 = \sqrt{-G}\,\frac{1}{2}H^2(\theta)\left(g_6^2(W^{+A}W^-_A) + (g_6^2 + g'^2_6)Z^{0\,A}Z^0_A\right),$$

where $H(\theta)$ is the configuration for the Higgs field [8], which can be approximated by a step of width θ_ϕ. If we denote $\int R^2 d\theta\sin\theta d\varphi H^2(\theta) = v^2/2$, the usual masses for gauge bosons arise as square root of corrections for eigenvalues,

$$m_w = \frac{gv}{2}\,,\qquad m_z = \sqrt{g^2+g'^2}\,\frac{v}{2}\,,$$

where g, $g' = g_6/\sqrt{4\pi}\,R$ and v are electroweak constants of the Standard Model. So, the result for masses of the gauge bosons is reproduced if

$$H(0) \sim \frac{v}{\sqrt{2\pi}\,R\theta_\Phi} \sim 10^3\ \mathrm{TeV}^2$$

at $R\sim 100\,\mathrm{TeV}$ and $\theta_\Phi \sim 0.01$.

The first correction to the profile of the Z-boson mode is

$$(g^2+g'^2)R^2\sum_{l=1}^{l=\infty}\frac{V_{0l}}{l(l+1)}\frac{1}{R}Y_{l0}(\theta)\,, \tag{C.1}$$

where

$$V_{0l}=\int R^2 d\theta\sin\theta d\varphi H^2(\theta)\frac{1}{\sqrt{4\pi}}Y_{l0}\sim v^2\left[\begin{array}{ll}\sqrt{l} & \text{at}\;\; l\theta_\Phi\le 1\,,\\ \mathrm{e}^{-lF(\theta\Phi)} & \text{at}\;\; l\theta_\Phi\gg 1\,.\end{array}\right.$$

The last approximation can be obtained in the same way as in appendix D. The value of V_{0l} is maximal at $l\sim 1/\theta_\Phi$, $V_{0l}/l(l+1)\sim 1/l^{3/2}$, and, therefore, the sum (C.1) is saturated by the lightest modes. Thus, the Z-boson mode recives small, of order of

$$\frac{1}{R}(g^2+g'^2)(Rv)^2\sim 10^{-4}g^2\ \text{at}\ R\sim 100\,\mathrm{TeV}\,,$$

θ-depending correction. At this level one could expect family non-universal couplings of the fermions with the Z boson. However, this non-universality is also generated due to an interchanging by non-zero modes of photons or (and) Z bosons. One can easily estimate the latter in a way similar to appendix D, and finds it at the level of

$$g\theta_A^2\sim 10^{-2}g\gg 10^{-4}g^2\,.$$

Therefore, we do not take into account the non-trivial profile of Z-boson mode and treat it as a constant.

D. Gauge-fermion lagrangian

Substituting eqs. (A.3), (A.2), (A.1), and (3.2) into eq. (3.1), one finds

$$\begin{aligned}J^{4,5}&=0\,,\\ J^\mu&=\sum_{m,n=1}^{3}\rho_{mn}\mathrm{e}^{i\varphi(m-n)}j^\mu_{mn}\,,\end{aligned}$$

where

$$\begin{aligned}\rho_{mn}&=\rho_{nm}=f_2(m)f_2(n)+f_3(m)f_3(n),\\ j^\mu_{mn}(x)&=(j_{nm})^*=a^\dagger_m\bar\sigma^\mu a_n=\bar\psi_m\gamma^\mu\frac{1+\gamma_5}{2}\psi_n\end{aligned}$$

and the four-dimensional Dirac matrices are

$$\gamma^\mu=\begin{pmatrix}0&\sigma^\mu\\ \bar\sigma^\mu&0\end{pmatrix},\qquad\gamma_5=\begin{pmatrix}1&0\\0&-1\end{pmatrix};$$

$\psi_n(x)=(a_n(x),b_n(x))^T$. We see that the zero fermionic modes do not interact with the scalar fields B_l, B_{lm}, while their interaction with the vector Kalutza-Klein tower is given by

$$\mathcal{L}_6=e_6\frac{\sqrt{-G}}{R}\sum_{n,m}^{3}\sum_{l=0}^{\infty}\sum_{k=-l}^{l}\rho_{mn}Y_{lk}(\theta,\varphi)\mathrm{e}^{i\varphi(m-n)}\cdot A^{l,k}_\mu(x)j^\mu_{mn}(x)$$

(hereafter, we neglect higher fermionic modes whose contributions to flavour violating processes are suppressed). In the effective four-dimensional theory,

$$\mathcal{L}_4 = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \mathcal{L}_6 = e \cdot \sum_{m,n}^{3} \sum_{l=0}^{\infty} (-1)^{\frac{(n-m)-|n-m|}{2}} E_{mn}^{l,m-n} A_\mu^{l,(n-m)}(x) j_{mn}^\mu(x) \,, \tag{D.1}$$

where

$$\begin{aligned} E_{mn}^{l,k} &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sqrt{-G}\, \rho_{mn} Q_l^{|k|} \mathrm{e}^{i\varphi(m-n+k)} \,, \\ E_{mn}^{l,n-m} = E_{nm}^{l,n-m} &= 2\pi \int_0^{\pi} d\theta \sqrt{-G}\, \rho_{mn} Q_l^{|n-m|} \,, \end{aligned} \tag{D.2}$$

$$Q_l^j = (-1)^j \sqrt{2l+1} \sqrt{\frac{(l-j)!}{(l+j)!}}\, P_l^j(\cos\theta) \,. \tag{D.3}$$

In eq. (D.1), integration over φ resulted in selection rules which mean, in particular, that the φ-independent zero mode of the vector field does not mediate flavour changing ($m \neq n$) processes.

The four-dimensional charge e is

$$e = \frac{e_6}{\sqrt{4\pi}\, R} \cdot E_{nn}^{0,0} = \frac{e_6}{\sqrt{4\pi}\, R} \cdot 2\pi \int_0^{\pi} d\theta \sqrt{-g}\, \rho_{nn} Q_0^0 = \frac{e_6}{\sqrt{4\pi}\, R} \,,$$

where we have used eqs. (3.3), (D.3).

To estimate $E_{mn}^{l,m-n}$, we note that f_i are localized in the region $\theta < \theta_A$, where θ_A is the size of the gauge field which forms the vortex. We are working in the regime

$$0 < \theta_\Phi < \theta_A < \theta_\Psi \ll 1 \,,$$

where $R\theta_\Phi$ is the size of the vortex on which fermions are localised and $(R\theta_\Psi)^{-1}$ is the energy scale of the fermionic non-zero modes (see ref. [8] for details). At $\theta \geq \theta_A$,

$$f_i(n) \sim \frac{1}{\theta^n} \mathrm{e}^{-\theta/\theta_\Psi} \,, \tag{D.4}$$

Therefore, the integral (D.2) is saturated at $\theta < \theta_A$, and we can use the behaviour of P_l^m at the origin (J_m is Bessel function of the first kind),

$$P_l^m(\cos\theta) = (-1)^m \left[\left(l + \frac{1}{2}\right) \cos\frac{\theta}{2}\right]^m J_m\left((2l+1)\sin\frac{\theta}{2}\right) + \mathcal{O}\left(\sin^2\frac{\theta}{2}\right) \sim (-1)^m l^m J_m(l\theta) \,,$$

to find

$$Q_l^m \sim \sqrt{l}\, J_m(l\theta) \simeq \left[\begin{array}{ll} l^{m+1/2}\theta^m & \text{at } l\theta_A \ll 1 \,, \\ \dfrac{\cos(l\theta - 1/2(m\pi) - 1/4\pi)}{\sqrt{\theta}} & \text{at } l\theta_A \gtrsim 1 \,. \end{array} \right. \tag{D.5}$$

At $l\theta_A \ll 1$, one obtains, from eqs. (D.5) and (D.2),

$$E_{mn}^{l,m-n} \sim l^{|m-n|+1/2} \int_0^{\pi} d\theta \sqrt{-G}\, \rho_{mn} \theta^{|m-n|} \,.$$

This integral can be estimated in the saddle point approximation. Indeed, $f_i(n)$ have a well localized maximum at $\theta \sim \theta_A$, so $\sqrt{-G}\,\rho_{mn}$ also has a maximum near this point. Using this fact and the normalization conditions (3.3) one finds

$$\int_0^\pi d\theta \sqrt{-G}\,\rho_{nm} \simeq 1$$

and, thus,

$$E_{mn}^{l,m-n} \sim l^{|m-n|+1/2}\theta_A^{|m-n|} \qquad \text{at } l\theta_A \ll 1\,.$$

The regime $l\theta_A \gtrsim 1$ contains two cases: 1) $l\theta_A \simeq 1$, 2) $l\theta_A \gg 1$. The first one can be worked out in the same way as the $l\theta_A \ll 1$ case. This is due to the fact that in eq. (D.5), the argument of cosine $l\theta_A \sim 1$. So, one has

$$E_{mn}^{l,m-n} \sim \frac{1}{\sqrt{\theta_A}} \qquad \text{at } l \simeq \frac{1}{\theta_A}\,.$$

In the regime $l\theta_A \gg 1$, however, the integral is not saturated at $\theta \sim \theta_A$, but rather (due to quick oscillations of cosine) at a complex value of θ — the pole of $\ln \rho_{mn}$. Since f_i have poles at the origin at $\theta_A \to 0$ (eq. (D.4)), the pole of $\ln \rho_{mn}$ should develop a non-zero imaginary part which tends to zero as $\theta_A \to 0$. So,

$$E_{mn}^{l,m-n} \sim \mathrm{e}^{-lF(\theta_A)} \qquad \text{at } l\theta_A \gg 1\,,$$

that is the couplings to highest modes are exponentially suppressed.

E. Rotation to physical states

The mass matrix (3.7) can be diagonalized by a biunitary transformation $S_d^\dagger M_D T_d = M_D^{\text{diag}}$. One can find S_d as an unitary matrix which diagonalizes $M_D M_D^\dagger$:

$$S_d^\dagger M_D M_D^\dagger S_d = M_D^{\text{diag}} (M_D^{\text{diag}})^\dagger$$

(the unitary matrix T_d obeys $T_d^\dagger M_D^\dagger M_D T_d = M_D^{\text{diag}}(M_D^{\text{diag}})^\dagger$). Matrices S_u, T_u diagonalize, in the same way, the mass matrix of up quarks M_U. According to refs. [4, 8], the mass matrices have the following form,

$$M_D \propto \begin{pmatrix} \delta^4 & \epsilon_d\delta^3 & 0 \\ 0 & \delta^2 & \epsilon_d\delta \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_U \propto \begin{pmatrix} \delta^4 & 0 & 0 \\ \epsilon_u\delta^3 & \delta^2 & 0 \\ 0 & \epsilon_u\delta & 1 \end{pmatrix},$$

where a real parameter $\delta \sim 0.1$ and two complex parameters ϵ_u, ϵ_d have absolute values of order 0.1. Then, up to the second order in ϵ,

$$S_d = \begin{pmatrix} 1 - \frac{|\epsilon_d|^2}{2}\alpha_d^2 & \epsilon_d\alpha_d & \epsilon_d^2\beta_d' \\ -\epsilon_d^*\alpha_d & 1 - \frac{|\epsilon_d|^2}{2}(\alpha_d^2+\gamma_d^2) & \epsilon_d\gamma_d \\ (\epsilon_d^*)^2\beta_d & -\epsilon_d^*\gamma_d & 1 - \frac{|\epsilon_d|^2}{2}\gamma_d^2 \end{pmatrix},$$

$$S_u = \begin{pmatrix} 1 - \frac{|\epsilon_u|^2}{2}\alpha_u^2 & \epsilon_u^*\alpha_u & (\epsilon_u^*)^2\beta_u' \\ -\epsilon_u\alpha_u & 1 - \frac{|\epsilon_u|^2}{2}(\alpha_u^2+\gamma_u^2) & \epsilon_u^*\gamma_u \\ \epsilon_u^2\beta_u & -\epsilon_u\gamma_u & 1 - \frac{|\epsilon_u|^2}{2}\gamma_u^2 \end{pmatrix},$$

where

$$\alpha_d = \frac{m_{22}m_{12}}{m_{22}^2 - m_{11}^2} \sim \delta\,, \qquad \gamma_d = \frac{m_{33}m_{23}}{m_{33}^2 - m_{22}^2} \sim \delta\,,$$
$$\alpha_u = \frac{m_{11}m_{12}}{m_{22}^2 - m_{11}^2} \sim \delta^3\,, \qquad \gamma_u = \frac{m_{22}m_{23}}{m_{33}^2 - m_{22}^2} \sim \delta^3\,;$$
$$\beta_d \sim \delta^2\,, \quad \beta_d' \sim \delta^6\,, \quad \beta_u \sim \delta^6\,, \quad \beta_u' \sim \delta^{10}\,.$$

To derive the dependence of the matrices T on ϵ and δ one can use the following trick: the matrix M_u transfers to $M_d^\dagger$ under replacement $\epsilon_u \to \epsilon_d^*$. This means that T_d have the same structure as S_u with replacement $\epsilon_u \to \epsilon_d^*$. From the expressions for S and T, one can see that $M_D^{\rm diag}$ and $M_U^{\rm diag}$ are real.

Let us consider now $U^{CKM} = S_u^\dagger S_d$. Unphysical phases can be removed by a transformation $F_u^\dagger U^{CKM} F_d$, where

$$F_d = \begin{pmatrix} {\rm e}^{i\phi_1^d} & 0 & 0 \\ 0 & {\rm e}^{i\phi_2^d} & 0 \\ 0 & 0 & {\rm e}^{i\phi_3^d} \end{pmatrix} \qquad F_u = \begin{pmatrix} {\rm e}^{i\phi_1^u} & 0 & 0 \\ 0 & {\rm e}^{i\phi_2^u} & 0 \\ 0 & 0 & {\rm e}^{i\phi_3^u} \end{pmatrix}.$$

However, S_u is "almost" diagonal. So, in the first approximation $U^{CKM} = S_d$ and all CKM phases can be removed by F_u, F_d with $\phi_i^u = \phi_i^d \equiv \phi_i$ for $i = 1, 2, 3$ and $\phi_1 - \phi_2 = \phi_2 - \phi_3 = \arg\epsilon_d \equiv \phi_d$. If we take into account S_u, $(\phi_1 - \phi_2)$ recives a small correction ($\phi_u \equiv \arg\epsilon_u$),

$$\phi_1 - \phi_2 = \phi_d + \left(\frac{\alpha_u}{\alpha_d}\right) \left|\frac{\epsilon_u}{\epsilon_d}\right| \sin(\phi_d + \phi_u)\,. \tag{E.1}$$

After these phase rotations and transformations to the mass basis for fermions, the matrices of gauge interactions become $F_d^\dagger S_d^\dagger A_\mu S_d F_d$ and $F_d^\dagger T_d^\dagger A_\mu T_d F_d$. Off-diagonal elements of T_d are smaller than the same elements of S_d, so the dominant flavour violating interactions are determined by the matrix $\tilde{\mathbf{A}}^\mu = S_d^\dagger \mathbf{A}^\mu S_d$:

$$\begin{pmatrix} \mathbf{A}_{11} - 2{\rm Re}(\epsilon^*\alpha\mathbf{A}_{12}) & \mathbf{A}_{12} + \epsilon\alpha(\mathbf{A}_{11} - \mathbf{A}_{22}) - \gamma\epsilon^*\mathbf{A}_{13} & \mathbf{A}_{13} + \epsilon(\gamma\mathbf{A}_{12} - \alpha\mathbf{A}_{23}) \\ \mathbf{A}_{12}^* + \epsilon^*\alpha(\mathbf{A}_{11} - \mathbf{A}_{22}) - \epsilon\gamma\mathbf{A}_{13}^*) & \mathbf{A}_{22} + 2{\rm Re}(\epsilon^*(\alpha\mathbf{A}_{12} - \gamma\mathbf{A}_{23})) & \mathbf{A}_{23} + \epsilon^*\alpha\mathbf{A}_{13} + \epsilon\gamma(\mathbf{A}_{22} - \mathbf{A}_{33})) \\ \mathbf{A}_{13}^* + \epsilon^*(\gamma\mathbf{A}_{12}^* - \alpha\mathbf{A}_{23}^*) & \mathbf{A}_{23}^* + \epsilon\alpha\mathbf{A}_{13}^* + \epsilon^*\gamma(\mathbf{A}_{22} - \mathbf{A}_{33}) & \mathbf{A}_{33} + 2{\rm Re}(\epsilon^*\gamma\mathbf{A}_{23}) \end{pmatrix}^\mu \tag{E.2}$$

(we denoted α_d (ϵ_d) as α (ϵ) for convenience and presented S_d in the first approximation on ϵ_d). After transformation $F_d^\dagger \tilde{A}_\mu F_d$, the element $\tilde{A}_{12}$ recives an additional phase ${\rm e}^{i(\phi_2 - \phi_1)}$, which is approximately opposite to phase of ϵ_d, eq. (E.1). It means that the phase of interaction through $(A_{11} - A_{12})$, which is responsible for CP-violation in kaons (see section 4.3) is suppressed in our model:

$$\arg(\epsilon_d {\rm e}^{i(\phi_2 - \phi_1)}) = -\left(\frac{\alpha_u}{\alpha_d}\right) \left|\frac{\epsilon_u}{\epsilon_d}\right| \sin(\phi_d + \phi_u)\,.$$

The consideration of U^{CKM} in second order in ϵ gives $U_{13}^{CKM} \sim \epsilon_u^*\epsilon_d^*\delta^4$, $U_{31}^{CKM} \sim (\epsilon_d^*)^2\delta^2$. So, the CKM matrix can not be made real by any rotations by F_u, F_d, so the CP-violation in our model arises in the same way as in the Standard Model.

F. Effective four-fermion interactions

One may calculate the effective charge,

$$g_{mn} = e^2 4\pi^2 \int_0^\pi \int_0^\pi d\theta d\theta' \sqrt{G(\theta)G(\theta')}\, \rho_{mn}(\theta)\rho_{mn}(\theta') \sum_{l=1}^{\infty} \frac{R^2}{l(l+1)} Q_l^{|n-m|}(\theta) Q_l^{|n-m|}(\theta')\,,$$

directly, by making use of the fact that, at $m > 0$,

$$\sum_{l=m}^{\infty} \frac{1}{l(l+1)} Q_l^m(\theta) Q_l^m(\theta') = -2G_m(\theta,\theta')\,,$$

where

$$\left(\frac{1}{\sin\theta}\partial_\theta \sin\theta \partial_\theta - \frac{m^2}{\sin^2\theta}\right) G_m(\theta,\theta') = \delta(\theta-\theta')\,.$$

Explicitly,

$$G_m(\theta,\theta') = -\frac{1}{2m}\left(\tan^m\frac{\theta}{2}\cot^m\frac{\theta'}{2}\Theta(\theta'-\theta) + \tan^m\frac{\theta'}{2}\cot^m\frac{\theta}{2}\Theta(\theta-\theta')\right).$$

At $m = 0$,

$$\sum_{l=1}^{\infty} \frac{1}{l(l+1)} Q_l^0(\theta) Q_l^0(\theta') = -2G_0(\theta,\theta')\,,$$

where

$$\frac{1}{\sin\theta}\partial_\theta \sin\theta \partial_\theta G(\theta,\theta') = \delta(\theta-\theta') - \frac{1}{2}$$

and

$$G(\theta,\theta') = \frac{1}{2} + \Theta(\theta'-\theta)\ln\left(\sin\frac{\theta'}{2}\cos\frac{\theta}{2}\right) + \Theta(\theta-\theta')\ln\left(\sin\frac{\theta}{2}\cos\frac{\theta'}{2}\right).$$

Then for $m \neq n$,

$$\begin{aligned} |g_{mn}| &< 8\pi^2 e^2 R^2 \left(\int_0^\pi d\theta \sqrt{-G}\,\rho_{mn} \tan^{|m-n|}(\theta)\right) \times \left(\int_0^\pi d\theta \sqrt{-G}\,\rho_{mn} \cot^{|m-n|}(\theta)\right) \\ &\sim e^2 R^2 \theta_A^{|m-n|} \times \theta_A^{-|m-n|} = e^2 R^2\,, \end{aligned}$$

which coincides with the naive estimate (3.8). For $m = n$, the result is the same (the dominant contribution comes from the constant term in $G_0(\theta,\theta')$).

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